STAT 821

Homework #4
Due on February 7, 2012 at noon

NOTE: The following problems are from Chapter 1 of the textbook.

1) Prove that in Example 6.25 (i) of the textbook, $T(X) = \left(X_{(1)},\ X_{(n)}\right)$ is not a complete statistic.

2) Let $f(x,\ y) = x^2 + y^2$, which maps from $\mathbb{R}^2$ to $(0,\ \infty)$. Show that f is convex function and describe the supporting hyper-plane $L(x)$ at the point (1, 0).

3) The logistic regression model is

$$p(y = \pm 1|x, w) = \frac{1}{1 + \exp(-y\ \cdot\ w^T x)}$$

where y is binary, x is a vector of covariates, and w is called the regression parameter. This model can be used for binary classification or for predicting the certainty of a binary outcome. To estimate w, the maximum likelihood estimation method is often used. The log-likelihood function $l(w)$ can be written as

$$l(w) = -\sum_{i=1}^{n} \log(1 + \exp(-y_i \cdot w^T x_i)).$$

The MLE is the maximize of $l(w)$ with respect to w. Assume xx^T is non-degenerated. Show that the MLE is unique if exists.

4) Let F and G be probability measures on $\mathbb{R}^k$. Let $h\colon \mathbb{R}^k \to [0, \infty)$ be convex with $h(x) \geq h(0) = 0$ for all $x \in \mathbb{R}^k$. Suppose that $E_F[h(X)] < E_G[h(X)]$. Show that there is a convex set $A \subset \mathbb{R}^k$ with $0 \in A$ such that $F(A) > G(A)$.
Hint: First try to prove that

$$E_F[h(X)] = -\int_{-\infty}^{\infty} h(x) dF(X > x) dx = \int_{0}^{\infty} F(h(X) > t) dt$$

5) Let $0 < \alpha < 1$.
(a) Show that the function $f(x, y) = -x^\alpha y^{1-\alpha}$ is convex on

$$\{(x, y) \in \mathbb{R}^2 : x > 0\ and\ y > 0\}$$

(b) for positive random variable X and Y with finite means, use the above result to prove the Hölder's inequality:

$$E[X^\alpha Y^{1-\alpha}] \leq [E(X)]^\alpha [E(Y)]^{1-\alpha}$$

(c) Use parts (a) and (b) to prove that the normalizing term $A(\eta)$ is a canonical exponential family is a convex function on the natural parameter space.